Principles and Standards

Representation: An Important Process for Teaching and Learning Mathematics

Francis (Skip) Fennell and Tom Rowan

Representation is more than a process; it is a way of teaching and learning mathematics.

Vignette: Four beginning third graders were working to solve a story problem: Anna had 63 stickers. She gave some to Tricia. Then Anna had 27 stickers. How many stickers did Anna give to Tricia? Brett said, "Well, 27 and 3 make 30, 30 more makes 60, and 3 more makes 63, so it's 3 + 30 + 3 = 36." Stacey wrote her solution on her paper as 60 – 20 = 40. She crossed out 40 and wrote 30 in its place; next she wrote 13 – 7 = 6, then the difference of 36. Heather, using Unifix cubes, gathered 6 ten-sized rods and 3 single cubes, took 2 away, then broke 1 of the remaining 4 rods into ones and took 7 away. She had 36 left. Maria added 4 to 63 to make 67, then added 4 to 27 to make 31. She said that seeing the difference of 36 was easy. These four students, all in the same grade at the same school, used different strategies and representations to solve the problem.

Francis (Skip) Fennell, ffennell@wmdc.edu, is a professor of education at Western Maryland College, Westminster, MD 21157. He has been directly involved in elementary mathematics education for over thirty years, serving as a teacher, supervisor, principal, writer, researcher, and policy and program analyst. Tom Rowan, trowan7@cs.com, is a part-time lecturer at the University of Maryland at College Park, Maryland, and an independent elementary mathematics consultant. His professional interests include working with preservice and in-service teachers to use reflective teaching and constructivism to help children learn mathematics. The authors have worked together in various capacities for over two decades.

Edited by **Jeane Joyner,** *jjoyner@dpi.state.nc.us, Department of Public Instruction, Raleigh, NC 27601, and* **Barbara Reys,** *reysb@missouri.edu, University of Missouri, Columbia, MO 65211. This department is designed to give teachers information and ideas for using the NCTM's* Principles and Standards for School Mathematics *(2000). Readers are encouraged to share their experiences related to* Principles and Standards *with* Teaching Children Mathematics. *Please send manuscripts to "Principles and Standards,"* TCM, *1906 Association Drive, Reston, VA 20191-9988.*

What Is Representation?

Representation is one of five mathematical processes presented in *Principles and Standards for School Mathematics* (NCTM 2000). These processes may be thought of as the "filter" through which the five Content Standards are developed. They should guide the ways in which we teach mathematics and support students' learning.

Teachers and students easily recognize the importance of the mathematics content topics in a particular grade or grade band but are less likely to recognize and understand the role of the mathematical processes. In the twelve years since the publication of *Curriculum and Evaluation Standards for School Mathematics* (NCTM 1989), curriculum developers, textbook publishers, and teachers have ignored the fact that content and process are intertwined. Students cannot solve problems without knowing the mathematical content conveyed in those problems, nor can they solve

FIGURE 1

The importance of representation

- Representations are powerful tools for thinking; they make mathematical ideas more concrete and available for reflection. They support and extend reasoning by helping students focus on essential features of the mathematical situation.
- Representations help students recognize the common mathematical elements of different situations.
- Understanding and using mathematical concepts and procedures are enhanced when students can transfer understanding among different representations of the same idea. Students need to develop and use a wide variety of representations.
- Teaching forms of representation as ends in themselves is not productive.
- Representations give learners useful tools for building understanding, communicating information, and demonstrating reasoning. (Greeno and Hall 1997)

problems without using mathematical processes.

The Process Standards provide essential support for learning mathematics content. The writers of *Principles and Standards* reaffirmed the importance of the four Process Standards—Problem Solving, Reasoning, Connections, and Communication—articulated in the NCTM's 1989 *Curriculum and Evaluation Standards*. They also acknowledge the central importance of the process of mathematical representation. In the 1989 *Standards* document, representation was discussed as part of the communication standard. Recognizing the important role that it plays as both a tool for communication and a tool for thinking (see **fig. 1**), representation is given more prominence in the updated *Principles and Standards*.

In the opening vignette of this article, the third graders used representations to model the action of a story problem that they understood. Their representations helped them solve the problem and share their thinking with others. The students used mental mathematics, paper and pencil, linking cubes, and pictures to represent their actions as they solved the problem. Each representation gave the student who used it a means for understanding and thinking through the problem. Such representations are essential in enabling children to analyze problems and find ways to solve them. Note that the representations used by the children clearly grew out of their own thinking. This aspect is an essential component of good representations—they should represent how the children are thinking about a problem. In some instances, children are taught to use concrete materials as the only way to solve a problem, and such materials may come to replace the child's thinking rather than represent it. As a result, the materials may actually interfere with learning or, at the least, become an alternative way of solving problems rather than a pathway to understanding mathematics.

When students are able to represent a problem or mathematical situation in a way that is meaningful to them, the problem or situation becomes more accessible. Using representations—whether drawings, mental images, concrete materials, or equations—helps students organize their thinking and try various approaches that may lead to a clearer understanding and a solution.

Students' thinking and the representations that express this thinking can vary greatly, even when addressing a single idea. One student may orally describe her interpretation of the mathematical concept or problem. Another may model it with base-ten blocks. Still another may draw a picture that illustrates an understanding and a solution for the problem, and another may use an application on the computer to represent and solve the problem. A computer representation could take the form of a geometric shape that is already in the computer software or one that is drawn and manipulated by the student. Students could also use animated manipulatives to model a situation.

The following conversation with a fourth-grade student illustrates another important point about representations, namely, that numerical representations, like concrete representations, should represent students' thinking.

Interviewer. Melanie, these two circles represent pies that were each cut into eight pieces for a party. This pie on the left had seven pieces eaten from it. How much pie is left there?

Melanie. One-eighth.

Interviewer. Could you write that number for me?

Statement of the Representation Standard

Instructional programs from prekindergarten through grade 12 should enable all students to—

- create and use representations to organize, record, and communicate mathematical ideas;
- select, apply, and translate among mathematical representations to solve problems;
- use representations to model and interpret physical, social, and mathematical phenomena.

Melanie. Yes. [Writes 1/8.]

Interviewer. The pie on the right had three pieces eaten from it. How much is left of that pie?

Melanie. Five-eighths.

Interviewer. Please write that number for me. [Melanie writes 5/8.] Melanie, if you put those together, how much of a pie is left?

Melanie. Six-eighths.

Interviewer. Write that for me. [Melanie writes 6/8.] Melanie, could you write a number sentence to show what you just did? If you want, you may use the numbers you already wrote.

[Melanie writes 1/8 + 5/8 = 6/16 by putting the plus sign and equals sign between the numbers she has already written and changing the 8 in 6/8 to 6/16.]

Interviewer. What is the answer?

Melanie. Six-sixteenths.

Interviewer. That's not the same as you told me before. Is that OK?

Melanie. Yes, this is the answer you get when you add the fractions.

What could have happened to allow Melanie to be able to deal with the story and its representations so well, yet cause her to make such an error when switching to numerical representations? Although we do not know for sure, we believe that Melanie made this mistake because the numbers and symbols that she used were not representations of her thinking. If the symbolic work that students do does not represent understanding, if it is not a way of communicating their thinking, then children may be willing to accept nonsense answers to mathematical problems simply because they arrived at the answers by manipulating symbols in a way that they thought was correct. We must always remember that the numbers that children write must be representations of their thinking. As soon as symbols become replacements for thinking, we have missed our learning target.

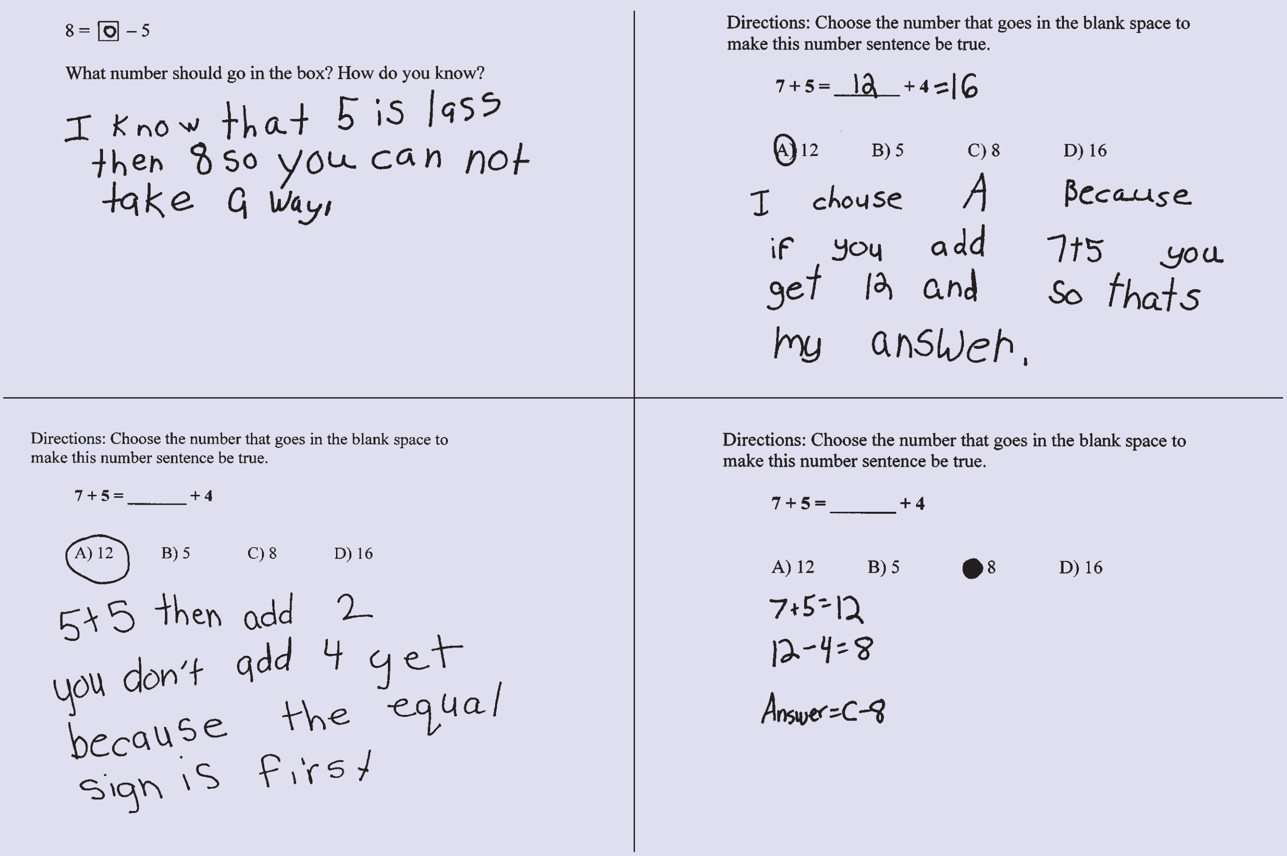
Promoting Appropriate Use of Representation in the Classroom

At the pre-K–6 levels, representation must be an important component in the daily planning of mathematics experiences for all children. Questions such as the following may drive such planning:

- How will my students represent the mathematical ideas of the lesson?

FIGURE 2

Students' interpretations of the = symbol



- What models might be helpful in representing such mathematical ideas?
- How might my students use representations to organize, record, and communicate mathematical ideas?
- How might my students select, apply, and translate mathematical representations to solve problems?
- How will my students use representations to model and interpret physical, social, and mathematical phenomena?
- What does the representation being used by a student tell me about that student's understanding of the mathematics?
- How can I encourage my students to use representations regularly to enable them to explore mathematical ideas?

When some children do not naturally turn to representations to clarify their ideas, teachers must find ways to help them get started. Other students can provide models. Many teachers circulate while students are working on a problem and select those students who are using good strategies and representations to share their work when the class comes together to discuss the problem. Helping students model their thinking is one way to give them access to representations that they can use to solve a problem (Steele 2000).

Students need to see, hear about, and try various representations for any problem. Teachers play an important role in modeling various representations by asking students what they are thinking, then representing that thinking with drawings, equations, or informal notes. Modeling appropriate representations of an idea is a good way to help students remember and visualize their ideas. Students must understand, however, that the teacher's representation is not the only correct way to represent the idea. The goal is to give students opportunities to think through, and choose from, a variety of representations in their quest to make sense of the mathematics that they are learning.

Illustrating Representation in the Mathematics Classroom

The following examples show how students use representations to model, explore, and better understand particular concepts and how teachers can learn about students' thinking from those representations.

Representing equality

Sometimes students do not understand the representations that adults use and take for granted. An example is the "is equal to" symbol. Typically, the "=" symbol comes after the operation symbol (e.g., 43 + 69 = 112) rather than before it (e.g., 112 = 43 + 69). This placement causes some students to misinterpret the symbol (Falkner et al. 1999). Students may interpret the equals sign as representing the instruction "Now do it." Others interpret the symbol to mean "The answer comes next." These limited interpretations may hamper the leap into later mathematics because advancing in mathematics requires a more complete understanding of equations and equality. Symbols represent mathematical ideas. Teachers must be sensitive to students' thinking and work with students to ensure that they interpret the symbols correctly. **Figure 2** shows various interpretations of the = symbol by students from several grades.

Representing perimeter and area

Fourth-grade students made a five-centimeter-by-seven-centimeter rectangle on centimeter grid paper. They found the distance around that shape and used the term *perimeter*. Then they drew a line from one corner of the rectangle along a diagonal to the opposite corner. They cut the rectangle along that line. Next they were asked to find all the new shapes that they could make by reassembling the two resulting triangles along matching sides. Finally, they were asked to decide whether all the new shapes had the same perimeter and area. By manipulating the triangles, students discovered a variety of new quadrilaterals and triangles. They investigated the perimeters of these new shapes and logically explained why the perimeters were different instead of the same. Some students then proclaimed, "The perimeters change, but the areas are all the same." In this example, the teacher was able to furnish a representation that the students could manipulate and use to discover some interesting geometric ideas for themselves.

Representing measurements

Third graders were given rulers and the task of measuring the length of their textbook and other objects. The rulers that they were given were a bit unusual, because they were "broken." Some of the rulers started at 3 rather than 0. Others started at 2 or 1. Some of the students ignored this discrepancy, lined up the rulers with one edge of the book, and recorded the number that lined up with the other edge. They recorded the length without question, even though they found an eight-inch book to have a measure of ten or even twelve inches. Others complained to their teacher that they could not make the measurement because the ruler was "broken." When told to figure out a way to use the rulers, many students could not do so. ☛

Why were many of these students stumped by this task? The answer may be that they never understood that the meaning of the number they attain when measuring length represents what they would have gotten if they had counted the inches one at a time. In the past, they may simply have used a standard ruler, matching it to the length being measured and recording a number that had no real conceptual meaning to them. Many students may not connect the concept of measurement as a way of counting units with the measurement tool, such as a ruler or formula, because they are not encouraged to represent and use measurement units in their most basic forms, in other words, to actually count the units.

The acts of representing measurement units and procedures are often skipped as students are pushed too quickly to use procedures that produce answers having no real meaning for them. For many students, the result is efficiency at the price of understanding. Note how fourth-grade students responded to the following item from the sixth National Assessment of Educational Progress (Kenney and Silver 1997):

> A rectangular carpet is 9 feet long and 6 feet wide. What is the area of the carpet in square feet?
>
> a) 15 b) 27
> c) 30 d) 54

Of the students tested, 45 percent chose (a) as the answer; 16 percent chose (b); 19 percent chose (c); and 19 percent chose (d), the correct response. We conclude that 81 percent of the students tested were unable to visualize, or represent, a nine-by-six rectangle and units of square feet. If they had been able to do so, then they could have used counting to solve the problem; they would not have needed any formula. Activities that involve students in measuring with single units; estimating measurements; and creating their own measurement tools, including formulas, by applying the single units can help students build the understanding that the number attained in a measurement situation is a representation of the attribute that they are measuring.

Representing data

Second graders were working on a question that asked, "How tall are second graders?" Initially, the students were confused by the question because they knew who was tallest and shortest in the class. One student suggested that the students line up according to their heights, forming a human bar graph. This idea seemed to work. The students could see not only who was tallest and shortest but also what the middle of the class looked like, and they could think about how tall most of the students were. They decided that if they counted from each end, they could find the middle height for the class. They could also spot the number of students who were very close to the same height. The students made the following statements about their real-life graph:

- Most of the class members were between the heights of Stacey and Heather.
- Brett was in the middle.
- Quinn was shortest, and Tom was tallest.
- More short people were in the class than tall people. (The students defined tall as being a bit taller than those in the middle. They defined short similarly.)

The students returned to their seats and wrote about the class data in their journals. Note that these students were analyzing measures of center, as well as the entire distribution (their class), in response to a question that interested them. They physically represented the median, mode, and range and informally discussed each of these important ideas. Their writing served as an additional representation of their thinking.

Summary

Representation is a process, an essential component of both teaching and learning, a way to model mathematics, and a way for students to show their thinking about mathematics. Teachers can use representation to clarify mathematical ideas to students, to access students' mathematical thinking, and to help students translate a mathematical idea into a form that they can mentally or physically manipulate to gain understanding.

References

Falkner, Karen P., Linda Levi, and Thomas P. Carpenter. "Children's Understanding of Equality: A Foundation for Algebra." *Teaching Children Mathematics* 6 (December 1999): 232–36.

Greeno, James G., and Roger B. Hall. "Practicing Representation: Learning with and about Representational Forms." *Phi Delta Kappan* 79 (January 1997): 361–67.

Kenney, Patricia A., and Edward A. Silver. *Results from the Sixth Mathematics Assessment of the National Assessment of Educational Progress (NAEP).* Reston, Va.: National Council of Teachers of Mathematics, 1997.

National Council of Teachers of Mathematics (NCTM). *Curriculum and Evaluation Standards for School Mathematics.* Reston, Va.: NCTM, 1989.

———. *Principles and Standards for School Mathematics.* Reston, Va.: NCTM, 2000.

Steele, Diana F. "Enthusiastic Voices from Young Mathematicians." *Teaching Children Mathematics* 6 (March 2000): 464–68. ▲